CREEPING NON-NEWTONIAN FLOW AROUND A ROTATING SPINDLE

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An approximate solution to the problem is based on a variational formulation and conformal mapping of the given configuration on a rectangle. The solution provides also the upper and the lower bound of the estimate of the torque. The approach is applicable to a broad class of geometrical configurations and rheological models of purely viscous non-Newtonian behaviour. As an example, an approximate solution is shown for a Bingham material in a system of concentric spheres.

Solutions to problems of hydrodynamics of non-Newtonian flows in the vicinity of rotating spindles of finite dimensions has found its practical use mainly for evaluation of data from rotational viscosimeters[1-4]. Certain applications may be also found in simulation of mixing by rotating impellers[5-7]. The most important result in both cases is the torque characteristics, *i.e.* the dependence of the torque, M, on a characteristic dimension of the spindle and the angular velocity, Ω, (preserving geometrical similarity of the configuration of the spindle and the vessel containing the rotating spindle) for given mechanical properties of the liquid.

Recently published experimental studies[8,9] indicate that even with a highly elastic behaviour of the tested liquids the torque characteristics in region of the creeping flow depends mostly on their viscosity function. Conversely one can assume that for a theoretical estimate of the torque under the creeping flow conditions the equations of motion for purely viscous liquids may be used with

$$\tau_{ij} = \eta(D)\, D_{ij}\,. \tag{1}$$

For a Newtonian creeping flow ($\eta(D) = \text{constant}$) around rotating spindles in an infinite liquid or a liquid confined in an axially symmetric fixed vessel of finite dimensions there are numerous exact explicit solutions available in the literature[10,11]. These solutions are mostly constructed by methods based on the theory of analytical functions and conformal mapping.

It is interesting to note that simple analytical solutions available for non-Newtonian creeping flows of power-law liquids in systems with spherical symmetry have been published only recently[1-3]. For other configurations than concentric spherical surfaces or for other non-linear models of purely viscous behaviour than the power--law model the problem becomes that of solving a non-linear elliptic equation with mixed boundary conditions. This class of problems to our knowledge has not been solved to date excepting the creeping flow of a Bingham material between two concentric spherical surfaces of close radii[12].

In view of the prospects of the portable rotational rheometers stemming mainly from the low costs of the instruments of the type of the Brookfield viscometei it seems useful to work out methods as exact as possible of transforming the torque characteristics into the viscometric data. In case that for a given configuration the general functional transforming the torque characteristic into the viscosity function (a known functional such as *e.g.* the one for coaxial cylinders) cannot be derived one has to resort to model courses of the torque characteristics. These are determined by solving the equation of motion for a chosen rheological model, usually the power-law model. With the present state of art, when the solutions even for the power-law model have been known only for configurations with spherical symmetry[1-3], it seems natural to apply approximate methods.

The approximate solution constructed in this work starts from an intuitive concept that the component of the gradient of angular velocity perpendicular to the confining surface is much larger than the parallel one not only on surface of the spindle and the vessel but rather in the whole volume of the rotating liquid. A conformal mapping of the spindle and the vessel on two parallel sides of a rectangle enables this concept to be formulated in general. On neglecting completely the component of the gradient parallel to the surface of the spindle and the vessel the original problem is simplified to a boundary value problem in a single independent variable. This approximate solution, which shall be referred to as the one-gradient one, offers a rather narrow upper and lower bounds for the estimates of the torque and represents thus a convenient starting point for formulating effective perturbation methods for searching higher-order approximations.

Natural Coordinates

Let us formulate the problem first in polar cylindrical coordinates (z, r, ϕ) oriented so as to make $r = 0$ the axis of rotational symmetry of the kinematic boundary conditions and thus the axis of rotational symmetry of the velocity field. The formulation will be confined to the creeping rotational steady flow of the liquids exhibiting in viscometric flows zero differences of normal stresses. With no sources or sinks and the boundary surfaces satisfying

$$v_r = 0\,, \quad v_z = 0\,, \tag{2}$$

the condition (*2*) is satisfied in the whole studied region and the velocity field can be represented by a single scalar function – the field of angular velocity around the axis of symmetry, $r = 0$, as

$$v_\phi / r = \omega(r, z) . \tag{3}$$

In view of the applications of practical importance we shall confine ourselves to the cases when the spindle and the vessel, and hence the field of angular velocity display plane symmetry. Let the plane of symmetry be at $z = 0$. The kinematic boundary conditions may then be formulated as

$$\omega = \Omega \quad \text{for} \quad (r, z) \in \Gamma_R , \quad \omega = 0 \quad \text{for} \quad (r, z) \in \Gamma_W , \tag{4a, b}$$

$$\frac{\partial \omega}{\partial z} = 0 \quad \text{for} \quad (r, z) \in \Gamma_M , \quad \frac{\partial \omega}{\partial r} = 0 \quad \text{for} \quad (r, z) \in \Gamma_A , \tag{4c, d}$$

where Γ_i are the contours in the first quandrant of the r, z plane, see Fig. 1.

In accord with the Riemann's theorem, between every two open simply connected regions there is a variety of conformal mappings represented by an analytical function with three adjustable parameters enabling to choose for instance three points in each region to be matched by transformation[13]. On admitting another degree of freedom by leaving one parameter unspecified characterizing the shape of the region (in our case for example the length of the rectangle on which the examined flow region is being mapped), there is just one conformal mapping between the two regions enabling four points chosen in advance (appearing in the same order on the boundaries of the region) to be matched.

Conformal mapping $(r, z) \to (\xi, \psi)$, transforming an open region $\mathscr{A}$, confined by a continuous closed curve $\Gamma_R \cup \Gamma_A \cup \Gamma_W \cup \Gamma_M$ in the coordinates (r, z), to a rectangle with the sides Γ_R, Γ_A, Γ_W, Γ_M in the coordinates (ξ, ψ), exists in accord with the above theorem and defines natural coordinates of the problem (ξ, ψ), Fig. 1. If the confining curves Γ_i are smooth in (r, z) and intersect at right angles, the corresponding conformal mapping is defined by a holomorphous analytical function on the closed region $\bar{\mathscr{A}}$.

The kinematic boundary conditions may be written in the natural coordinates of the problem according to Eqs (4) as

$$\omega = \Omega \quad \text{for} \quad \Gamma_R , \quad \xi = \xi_R , \quad \psi_M < \psi < \psi_A \tag{5a}$$

$$\omega = 0 \quad \text{for} \quad \Gamma_W , \quad \xi = \xi_W , \quad \psi_M < \psi < \psi_A , \tag{5b}$$

$$\frac{\partial \omega}{\partial \psi} = 0 \quad \text{for} \quad \Gamma_M , \quad \psi = \psi_M , \quad \xi_R < \xi < \xi_W , \tag{5c}$$

$$\frac{\partial \omega}{\partial \psi} = 0 \quad \text{for} \quad \Gamma_A , \quad \psi = \psi_A , \quad \xi_R < \xi < \xi_W . \tag{5d}$$

Rheology and Model of the Flow

The rheological model of the Generalized Newtonian Liquid (GNL), Eq. (*1*), contains a single scalar material function. Apart from the basic form $\eta(D)$ this function may be written in a number of equivalent forms to be used subsequently without special emphasis: In addition to the common forms $\tau = \tau(D)$, or the inverse form $D = D(\tau)$ we have Bird's energy function[14]

$$U(D) = \int_0^D \tau(D)\,\mathrm{d}D \qquad (6a)$$

and the local flow index

$$n' = n'(D) = \frac{\mathrm{d}\ln\tau(D)}{\mathrm{d}\ln D}\,. \qquad (6b)$$

For a complete description of the course of the viscosity function it is of course necessary to furnish the data regarding the integration constant for Eq. (*6b*).

From general variational formulations of the equations of motion for the steady flow of the GNL (ref.[14,15]) one can easily derive the functional for the case under consideration

$$\mathrm{J}[\omega] = \iint_{\mathscr{A}} U(D)\,.\,r\,.\,\mathrm{d}r\,.\,\mathrm{d}z\,, \qquad (7)$$

to be minimized by the solution of the problem with the boundary conditions (*4*). As the transformation into the natural coordinates of the problem (ξ, ψ) is a con-

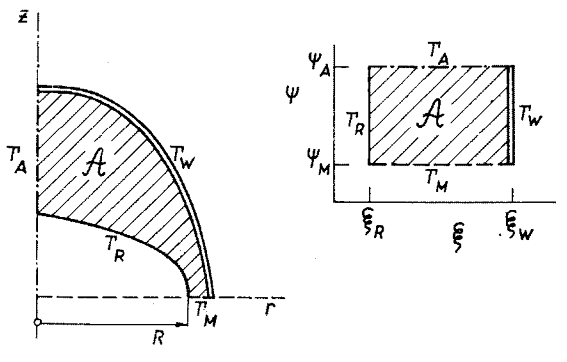

FIG. 1

Geometrical Configuration

Γ_A axis of rotation, Γ_M plane of symmetry, Γ_R surface of spindle, Γ_W surface of wall.

formal one, both Lamé's coefficients are identical and the non-zero components of the rate of deformation, can be expressed in the form

$$D_{\phi\xi} = -g^{-1} \cdot \frac{\partial \omega}{\partial \xi}, \quad D_{\phi\psi} = -g^{-1} \cdot \frac{\partial \omega}{\partial \psi}, \tag{8a, b}$$

where g is the normalized Lamé's coefficient

$$g(\xi, \psi) = \left[\left(\frac{\partial \ln(r)}{\partial \xi}\right)^2 + \left(\frac{\partial \ln(r)}{\partial \psi}\right)^2\right]^{1/2}. \tag{9}$$

The functional in Eq. (7) may be written in the natural coordinates as a two--dimensional integral over the rectangle $\mathscr{A} = (\xi_R, \xi_W) \times (\psi_M, \psi_A)$

$$\mathrm{J}[\omega] = \iint_{\mathscr{A}} U(D)\, r^3 \,.\, g^2 \,.\, \mathrm{d}\xi \,.\, \mathrm{d}\psi\,, \tag{10}$$

where the deformation rate is given by

$$D = \left[D_{\phi\xi}^2 + D_{\phi\psi}^2\right]^{1/2}. \tag{11}$$

The first and the second variation of the functional (*10*) may be written as

$$\delta\mathrm{J} = \iint_{\mathscr{A}} \eta\, r^3 \,.\, P \,.\, \mathrm{d}\xi \,.\, \mathrm{d}\psi = \iint_{\mathscr{A}} \varepsilon\, \Delta[\omega] \,.\, \mathrm{d}\xi \,.\, \mathrm{d}\psi \tag{12a, b}$$

$$\delta^2\mathrm{J} = \iint_{\mathscr{A}} \eta\, r^3 \,.\, (n' \,.\, P^2/S + N^2/S) \,.\, \mathrm{d}\xi \,.\, \mathrm{d}\psi\,, \tag{13}$$

where

$$\varepsilon(\xi, \psi) = \delta\omega \tag{14}$$

is the variation of the angular velocity and Δ is an elliptic differential operator of the problem:

$$\Delta[\omega] = \frac{\partial}{\partial \xi}\left(-\eta(D) \,.\, r^3 \,.\, \frac{\partial \omega}{\partial \xi}\right) + \frac{\partial}{\partial \psi}\left(-\eta(D) \,.\, r^3 \frac{\partial \omega}{\partial \psi}\right). \tag{15}$$

P, N, S are auxiliar quantities introduced by the relations

$$P(\omega, \varepsilon) = \frac{\partial \omega}{\partial \xi} \frac{\partial \varepsilon}{\partial \xi} + \frac{\partial \omega}{\partial \psi} \frac{\partial \varepsilon}{\partial \psi}, \tag{16a}$$

$$N(\omega, \varepsilon) = \frac{\partial \omega}{\partial \psi} \frac{\partial \varepsilon}{\partial \xi} - \frac{\partial \omega}{\partial \xi} \frac{\partial \varepsilon}{\partial \psi}, \tag{16b}$$

$$S(\omega) = \left(\frac{\partial \omega}{\partial \xi}\right)^2 + \left(\frac{\partial \omega}{\partial \psi}\right)^2 = g^2 D^2 . \tag{16c}$$

A sufficient condition for the functional (*10*) to be stationary, $\delta J = 0$, is clearly

$$\Delta[\omega] = 0 , \tag{17}$$

which is the equation of motion of a creeping rotational flow in the current form. From the structure of the second variation, Eq. (*13*), it is apparent that a sufficient condition for the stationary point (solution) of the functional (*10*) to be a global minimum is $n' \geqq 0$. This is satisfied in all cases of practical interest.

With the boundary conditions (*5*) the acceptable variations ε are constrained by the following homogeneous conditions

$$\varepsilon = 0 \quad \text{for} \quad \xi = \xi_R , \quad \xi = \xi_W , \tag{18a,b}$$

and

$$\frac{\partial \varepsilon}{\partial \psi} = 0 \quad \text{for} \quad \psi = \psi_M , \quad \psi = \psi_A . \tag{18c,d}$$

In the following the symbol ε (eventually subscripted) will be used to designate a class of sufficiently smooth functions satisfying the homogeneous boundary conditions (*18*).

One-Gradient Solution

The character of the boundary conditions suggests that the dependence of ω on the longitudinal coordinate ψ is far weaker than the dependence on the radial coordinate ξ:

$$\left|\frac{\partial \omega}{\partial \psi}\right| \ll \left|\frac{\partial \omega}{\partial \xi}\right| . \tag{19}$$

If the inequality (*19*) is satisfied, the argument of the scalar function (*6a*) may be replaced approximately by the value of the radial component of the rate of deformation:

$$U(D) \approx U(D_{\phi\xi}) . \tag{20}$$

Neglecting the longitudinal component $D_{\phi\psi}$ thus leads to the formulation of a "one-

-gradient" functional as

$$\mathrm{J}_1[\omega] = \iint_{\mathscr{A}} U(D_{\phi\xi})\, r^3 g^2 \,.\, \mathrm{d}\xi \,.\, \mathrm{d}\psi \tag{21}$$

which for an arbitrary ω approximates the principal functional J from below

$$\mathrm{J}_1[\omega] \leqq \mathrm{J}[\omega]\,. \tag{22}$$

The one-gradient approximation ω^1 of the exact solution ω_e we define as a function minimizing for the boundary conditions (5) the functional J_1. The first and the second variation of the functional (21) can be written in the form

$$\delta \mathrm{J}_1 = \iint_{\mathscr{A}} \varepsilon \frac{\partial}{\partial \xi} \left[\tau(D_{\phi\xi})\, r^3 g\right] . \, \mathrm{d}\xi \,.\, \mathrm{d}\psi\,, \tag{23}$$

$$\delta^2 \mathrm{J}_1 = \iint_{\mathscr{A}} n'(D_{\phi\xi}) \,.\, \eta(D_{\phi\xi}) \,.\, r^3 \,.\, \left(\frac{\partial \varepsilon}{\partial \xi}\right)^2 .\, \mathrm{d}\xi \,.\, \mathrm{d}\psi\,. \tag{24}$$

Provided that such solution exists, the corresponding stationary point of the functional (21) is according to (24) its minimum. As follows from Eq. (23), $\omega^1(\xi, \psi)$ is a solution of an ordinary differential equation with the boundary conditions (5a, b) while the longitudinal coordinate is a parameter of the problem. The solution may be written explicitly as

$$\omega^1 = \int_{\xi}^{\xi_W} g \,.\, D(C \,.\, r^{-3} \,.\, g^{-1}) \,.\, \mathrm{d}\xi\,, \tag{25}$$

where $C = C(\psi)$ is for each ψ a root of the equation

$$\Omega = \int_{\xi_R}^{\xi_W} g \,.\, D(C \,.\, r^{-3} \,.\, g^{-1}) \,.\, \mathrm{d}\xi\,. \tag{26}$$

As a proof of existence of the solution we need to prove that ω^1, defined by Eqs (25) and (26), satisfies also the boundary conditions (5c, d). For this purpose one can make use of the orthogonality of the functions $r(\xi, \psi)$, $z(\xi, \psi)$ ensured by the conformal mapping.

For brevity we shall present only the principal idea of the proof: The orthogonality of the functions $z(\xi, \psi)$, $r(\xi, \psi)$ ensures that in the neighbourhood of some contour ψ_0 = constant, where r = constant or z = constant, the following asymptotic relations hold

$$r(\xi, \psi_0 + \delta) \approx r_0(\xi) \,.\, (1 + 0(\delta^2))\,, \tag{27a}$$

$$g(\xi, \psi_0 + \delta) \approx g_0(\xi) \,.\, (1 + 0(\delta^2))\,, \tag{27b}$$

where

$$\delta = \psi - \psi_0 \ll 1\,. \tag{27c}$$

The viscosity function $D(\tau)$ in a sufficiently close neighbourhood of the line ξ_0 may be expressed using logarithmic derivatives as the following expansion

$$D(\xi, \psi_0 + \delta) \approx D(C_0 \,.\, r^{-3} \,.\, g_0^{-1}) \,.\, [1 + (1/n') \,.\, (a(\xi) \,.\, \delta + B(\xi)\, \delta^2)\,. \tag{28}$$

Substituting the last result into Eq. (*26*), which must hold asymptotically for all $\delta \to 0$ and comparing the quantities of equal order in δ one arrives at the relation $C(\psi_0 + \delta) = C_0 \,.\, (1 + 0(\delta^2))$. Thus also

$$\omega^1(\xi, \psi_0 + \delta) = \int_{\xi}^{\xi_W} g_0 \,.\, D(C_0 \,.\, r_0^{-3} \,.\, g_0^{-1}) \,.\, \mathrm{d}\xi \,.\, (1 + 0(\delta^2))\,. \tag{29}$$

This is essentially the sought result because clearly for $\psi \to \psi_0$ we have $\partial\omega^1/\partial\psi = 0(\delta)$ and the boundary conditions (*5c, d*) are met. The proof for $\psi_0 = \psi_A$, *i.e.* for $r \to 0$, $g \to \infty$, $D \to 0$ calls for certain modification in that r_0, g_0 are expressed in the form $f(\xi) \,.\, \delta^k$ and with the assumption of the existence of the limit n' for $D \to 0$, $D(\tau)$ is expressed in the corresponding power form.

As the exact solution ω_e represents the minimum of the functional J, estimation of the upper bound for the approximate methods poses no problem. The significance of the one-gradient approximation rests in that it furnishes also the lower bound for J. Clearly, for each approximate solution ω we have that $J_1[\omega] \leqq J[\omega]$. However, $J_1[\omega] \leqq J[\omega_e]$ is not generally true. Since we have $J_1[\omega_e] \leqq J[\omega_e]$ and the one-gradient solution represents the minimum of the functional $J_1[\omega]$, *i.e.* $J_1[\omega^1] \leqq$ $\leqq J_1[\omega]$ for each ω, the following inequalities are valid generally

$$J_1[\omega^1] \leqq J[\omega_e] \leqq J[\omega^1]\,. \tag{30}$$

The Determination of Dissipation

The data on the torque M and the dissipation of mechanical energy into heat, Q, may be regarded as equivalent owing to the relation $M = Q/\Omega$. In the following we shall therefore confine ourselves to the determination of the dissipation the approximate estimates of which for a given approximation of the angular velocity may be again defined as the corresponding functional. For instance

$$Q/(4\pi) = \mathrm{E}[\omega] = \int_{\mathscr{A}} \eta\, r^3 \,.\, S \,.\, \mathrm{d}\xi \,.\, \mathrm{d}\psi \,. \tag{31}$$

The first variation may be written in the form

$$\delta \mathrm{E} = \delta_S \mathrm{E} + \delta_\eta \mathrm{E}\,, \tag{32}$$

where

$$\delta_S \mathrm{E} = 2\,\delta \mathrm{J} \tag{33a}$$

and

$$\delta_\eta \mathrm{E} = -\iint_{\mathscr{A}} (1 - n') \,.\, \eta\, r^3 \,.\, P \,.\, \mathrm{d}\xi \,.\, \mathrm{d}\psi \,. \tag{33b}$$

The term $\delta_S \mathrm{E}$ expresses the effect of changing velocity gradient for a fixed viscosity field on the total change E. $\delta_\eta \mathrm{E}$ represents in contrast the changes of E brought about by the changes of the viscosity field. From the expression for the second variation E (not presented here) it follows that under the asumption $0 \leqq n' \leqq 1$ and for sufficiently smooth course of the viscosity function (which are the only cases of practical interest) $\delta^2 \mathrm{E} > 0$. Consequently, $\mathrm{E}[\omega]$ exhibits a sharp global minimum. Generally, this minimum occurs at a different point $\omega = \omega_\mathrm{E}$ than the point ω_e of the minimum of the principal functional of the problem, $\mathrm{J}[\omega]$. For an approximate determination of E one has therefore to search for a sequence of estimates ω_i ensuring that the corresponding sequence of $\mathrm{J}_\mathrm{i} = \mathrm{J}[\omega_\mathrm{i}]$ is non-increasing and calculate the corresponding $\mathrm{E}_\mathrm{i} = \mathrm{E}[\omega_\mathrm{i}]$. The criterion of convergence is the behaviour of the sequence J_i.

However, there is a strong relation between the quantities E and J enabling one of the direct variational methods to be used for construction of the minimizing sequence J_i and to obtain corresponding estimates of E_i without additional calculations.

Let us introduce the following functional of three independent arguments f_1, f_2, w:

$$(f_1, f_2)_\mathrm{w} = \iint_{\mathscr{A}} \eta(D_\mathrm{w}) \,.\, r^3 \left\{\frac{\partial f_1}{\partial \xi}\frac{\partial f_2}{\partial \xi} + \frac{\partial f_1}{\partial \psi}\frac{\partial f_2}{\partial \psi}\right\} \mathrm{d}\xi \,.\, \mathrm{d}\psi\,, \tag{34}$$

where

$$D_\mathrm{w} = g^{-1}\left[\left(\frac{\partial w}{\partial \xi}\right)^2 + \left(\frac{\partial w}{\partial \psi}\right)^2\right]^{1/2}. \tag{35}$$

The functional has the properties of a scalar product of the functions f_1, f_2 on a linear space which is a union of $\{\varepsilon\}$ and $\{\omega\}$ and permits one to introduce the norm

$$\|f\|_\mathrm{w} = (f, f)_\mathrm{w}^{1/2}\,. \tag{36}$$

The introduced scalar product (*34*) and the norm (*36*) containing the parametric function w have in view of the variational formulation of the approximate solution of the problem the following useful formal properties:

1) The exact solution ω_e and an arbitrary admisible ε satisfy the equation

$$(\omega_e, \varepsilon)_{\omega_e} = 0\,, \tag{37}$$

which is identical to Eq. (*12a*). *2*) The exact value of $E_e = E[\omega_e]$ is given by the expression

$$E_e = \|\omega_e\|^2_{\omega_e}\,. \tag{38}$$

The last equation is again identical to Eq. (*31*) for $\omega = \omega_e$. *3*) Ritz' method: Let us have a fixed first estimate of ω_0 and a set of trial functions ε_j. The approximate solution in the form

$$\omega_a = \omega_0 + \sum_j a_j \varepsilon_j \tag{39}$$

represents a local minimum on the set ε_j if the coefficients a_j are solutions of the following nonlinear set of equations ($j = 1, \ldots$) as

$$0 = (\omega_a, \varepsilon_j)_{\omega_a} = (\omega_0, \varepsilon_j)_{\omega_a} + \sum_i a_i (\varepsilon_i, \varepsilon_j)_{\omega_a}\,. \tag{40}$$

4) The approximate estimate of E corresponding to the approximation (*39*) may be expressed from Eq. (*31*) and on the basis of general properties of scalar products in the form

$$E_a = \|\omega_a\|^2_{\omega_a} = \|\omega_0\|^2_{\omega_a} - \|\sum_i a_i \varepsilon_i\|^2_{\omega_a}\,. \tag{41}$$

As it is common for nonlinear problems the set of trial functions and the convergence of the method in the sense of minimizing $J[\omega]$ must be studied for each particular case separately. However, as a rule the one-gradient approximation ω^1 ensures a good first estimate permitting a minimizing sequence to be constructed by making use of the following quasi-linearization based on replacing $\|\ \|_{\omega_a}$ norms by the $\|\ \|_{\omega_0}$ norms.

Let us start from a countable base ε_i and a first estimate $\omega_0 = \omega^1$. The ($k + 1$)-st estimate is defined as

$$\omega_{k+1} = \omega_k + \sum_{j=1}^{k} a_j^k \varepsilon_j\,, \tag{42}$$

where the coefficients a_j^k are solutions of the following linear system:

$$(\omega_{k+1}, \varepsilon_j)_{\omega_k} = 0\,, \quad j = 1, \ldots, k\,. \tag{43}$$

Corresponding sequence J_k is given by the relations

$$J_{k+1} = J_k + \delta J_k = J_k + \sum_{j=1}^{k} a_j^k (\omega_k, \varepsilon_j)_{\omega_k}, \tag{44a}$$

$$J_{k+1} - J_k = - \left\| \sum_{j=1}^{k} a_j^k \varepsilon_j \right\|_{\omega_k}^2 \leqq 0, \tag{44b}$$

and as such it is non-increasing. The sequence E_k is calculated from intermediate results

$$E_{k+1} \approx \|\omega_{k+1}\|_{\omega_k}^2 = \|\omega_k\|_{\omega_k}^2 - \left\| \sum_{j=1}^{k} a_j^k \varepsilon_j \right\|_{\omega_k}^2. \tag{45}$$

The described quasi-linearization clearly permits construction of a convergent minimizing sequence ω_k provided that the base ε_i is complete. With regard to the relation

$$\|\omega_e + \varepsilon\|_{\omega_e} = \|\omega_e\|_{\omega_e} (1 + O(\|\varepsilon\|_{\omega_e}^2)) \tag{46}$$

which is met for each ε (see Eq. (*33a, b*)) one may expect the sequence E_k to converge sufficiently rapidly toward E_e. In order to speed up convergence it is useful to utilize the first, *i.e.* the one-gradient approximation to arrange the base ε_i by the following rule

$$(\omega^1, \varepsilon_i)_{\omega^1}^2 > (\omega^1, \varepsilon_j)_{\omega^1}^2 \Rightarrow i < j. \tag{47}$$

For the special case n' = constant (power-law) we have precisely

$$E = (1 + n')\, J \tag{48}$$

and the problem may be formulated directly as one of a minimum dissipation. However, even in cases when Eq. (*48*) does not hold accurately the one-gradient estimate E_1 of the dissipation may serve at least qualitatively as the lower estimate. The one-gradient (lower) estimate of the dissipation for the one-gradient solution $E_1(\omega^1]$ may be expressed as a line integral

$$E_1[\omega^1] = \iint_{\mathscr{A}} \eta(D_{\phi\xi}) \,.\, D_{\phi\xi}^2 \,.\, r^3 \,.\, g^2 \,.\, d\xi \, d\psi = \tag{49a}$$

$$= \int_{\psi_M}^{\psi_A} C(\psi) \,.\, d\psi \tag{49b}$$

where $C(\psi)$ is an integration parameter from Eq. (*26*).

An Example of the One-Gradient Approximation

There exist cases when the one-gradient approximation is identical to the exact solution. For this it suffices that the one-gradient solution satisfy $(\partial\omega/\partial\psi) = 0$ over the whole $\mathscr{A}$. An example is the flow of a Newtonian liquid in a system of confocal spheroids and the flow of a power-law liquid in a system of concentric spheres. On the other hand, as an example of the situation when the one-gradient solution is not exact, one can put forth a sphere rotating in an infinite viscoplastic liquid. The latter problem has been solved by approximate methods earlier[12]. Neverthless, an order of magnitude estimate of the longitudinal derivatives, $\partial\omega/\partial\psi$, and corresponding estimate of dissipation in this case will reveal that the one-gradient solution, in this case not exact, still offers rather accurate estimates of the dissipation and the torque.

The Shvedov–Bingham model of the viscosity function may be written in the form

$$D[\tau] = \begin{cases} 0; & |\tau| \leqq \tau_0 \\ \dfrac{\tau - \tau_0}{\mu_B}; & |\tau| \geqq \tau_0 . \end{cases} \tag{50}$$

For a system of concentric spheres the conformal mapping onto a rectangle in (ξ, ψ) may be constructed *e.g.* as

$$r = R \,.\, \exp(\xi) \,.\, \cos(\psi)\,, \tag{51a}$$

$$z = R \,.\, \exp(\xi) \,.\, \sin(\psi)\,, \tag{51b}$$

$$g = \cos^{-1}(\psi)\,, \tag{51c}$$

which identifies $\xi_R = 0$ with the surface of a sphere of radius R and it is assumed that $\xi_w = \infty$. Substituting (*50*), (*51*) into the general Eqs (*25*), (*26*) and (*49b*) one arrives at the relations

$$w = \frac{\omega^1}{\Omega} = \frac{\exp(3\xi_0 - 3\xi) - 1 - 3\xi_0 + 3\xi}{3B\cos(\psi)}\,, \tag{52}$$

$$A > A_1(B) = \frac{M_1}{4\pi R^3 \tau_0} = \int_0^{\pi/2} \exp(3\xi_0) \,.\, \cos^2(\psi) \,.\, d\psi\,, \tag{53}$$

where we introduced instead of $C(\psi)$ an integration parameter $\xi_0(\psi)$, $C(\psi) = R^3 \,.\, \tau_0 \,.\, \cos^2(\psi) \,.\, \exp(3\xi_0(\psi))$ which for a given ψ is the solution of

$$3B\cos(\psi) = \exp(3\xi_0) - 3\xi_0 - 1 \tag{54}$$

and B is a dimensionless macroscopic parameter of the problem given by

$$B = \Omega \, . \, \tau_0/\mu_B \, . \tag{55}$$

The parameter ξ_0 has a definite physical meaning: $\xi = \xi_0(\psi)$ is the equation of a rotational surface confining the region of the flow. For $\xi \geqq \xi_0$ and a given $\psi =$ constant we have $D = 0$ (Fig. 2). In region $0 \leqq \xi \leqq \xi_0$ the field of radial components of the shear stress in the one-gradient approximation is given by

$$\tau_{r\phi} = \tau_0 \, . \, \exp\left(3\xi_0 - 3\xi\right) . \tag{56}$$

In order that we may construct the upper estimate $A(B)$ one has to know the derivative

$$\frac{\partial w}{\partial \psi} = \frac{\sin(\psi)}{B \, . \cos^2(\psi)} \, . \left(\xi_0 \, . \, \frac{1 - \exp(-3\xi)}{1 - \exp(-3\xi_0)} - \xi\right) . \tag{57}$$

From the last equation it is apparent that the longitudinal derivative vanishes not only on the bounding surfaces and the surfaces of symmetry but also on the boundaries of the flow region, $\xi = \xi_0$.

Local dissipation may be expressed in the form $\eta \, . \, D^2$. Its one-gradient estimate will be expressed as $\eta_1 \, . \, D_1^2$. Let us introduce the parameter q as a ratio of the longitudinal and radial derivative

$$q = \left|\frac{\partial w}{\partial \psi}\right| \Big/ \left|\frac{\partial w}{\partial \xi}\right| . \tag{58}$$

The upper bound for the estimate of local intensity of dissipation based on the one--gradient solution may then be expressed in the form $\eta_1 \, . \, D_1^2 \, . \, (1 + q^2)$. In the last expression we have made use of the fact that $\eta(D)$ is a non-increasing function of D. Denoting a suitable mean value of q for $\psi =$ constant by $q_2(\psi)$ the upper bound for the estimate may be expressed as

$$A \leqq A_2(B) = A_1(B) + \Delta A(B) \tag{59a}$$

where

$$\Delta A(B) = \int_0^{\pi/2} \exp\left(3\xi_0\right) . \cos^2(\psi) \, . \, q_2^2 \, . \, \mathrm{d}\psi \, . \tag{59b}$$

We shall show that for $B \leqq 1$ and $B \geqq 1$ Eqs (53) and (59) respectively enable us to construct asymptotic expansions for the lower and upper estimates of negligible difference.

The asymptotic expansions from Eq. (*54*) may be written as

$$\exp(3\xi_0) \approx 1 + (2b)^{1/2} + \frac{1}{3}(2b) + \frac{5}{72}(2b)^{3/2} + 0(b^2) \quad \text{for} \quad B \ll 1, \qquad (60a)$$

$$\exp(3\xi_0) \approx b + \ln(b) + 1 + \frac{\ln(b)}{b} + 0(b^{-1}) \quad \text{for} \quad B \gg 1, \qquad (60b)$$

where $b = 3B \,.\, \cos(\psi)$.
The lower bounds according to Eq. (*53*) may be expressed in the forms

$$A_1(B) \approx \int_0^{\pi/2} (\cos^2(\psi) + 6B \,.\, \cos^{5/2}(\psi) + 2B\cos^3(\psi) +$$

$$+ \frac{5}{72}(6B)^{3/2}\cos^{7/2}(\psi))\,d\psi + 0(B^2) = 0{\cdot}785 + 1{\cdot}762B^{1/1} + 1{\cdot}333B + 0{\cdot}637B^{3/2}$$

$$\text{for pro} \quad B \gg 1 \qquad (61a)$$

$$A_1(B) \approx \int_0^{\pi/2} (3B\cos^3(\psi) + (\ln(3B) + 1)\,.\,\cos^2(\psi) + \frac{\ln(3B)}{3B}\,.\,\cos(\psi) +$$

$$+ \ln(\cos(\psi))\,.\,\cos^2(\psi))\,d\psi + 0(B^{-1}) = 2B + 0{\cdot}785\,.\ln(3B) - 0{\cdot}152 + \frac{\ln(3B)}{3B}$$

$$\text{for} \quad B \ll 1 \qquad (61b)$$

shown also graphically in Fig. 3.

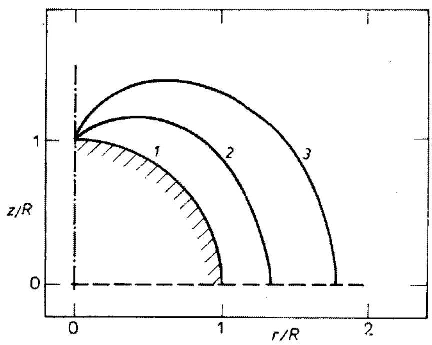

FIG. 2

Contours of Constant Angular Velocity for Rotation of a Sphere in a Viscoplastic Material, $B = 1$

1 $w = 1$ (surface of sphere), *2* $w = 0{\cdot}1$, *3* $w = 0$ (boundary of flow region).

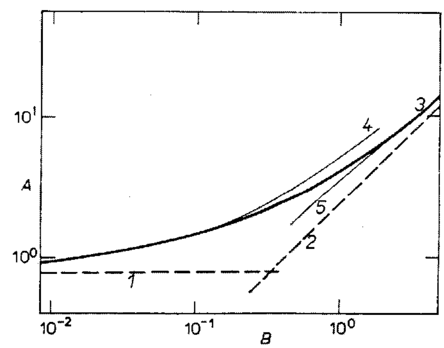

FIG. 3

Momentum Characteristic for Rotation of a Sphere in a Viscoplastic Material

1 Ideal plastic, *2* Newtonian liquid, *3* one--gradient estimate, *4* asymptote for $B \ll 1$, Eq. (*61a*), *5* asymptote for $B \gg 1$, Eq. (*61b*).

The estimate q_2 is constructed as a ratio of the maximum value of the longitudinal and radial derivative. The maximum value of the longitudinal derivative is obtained in the familiar way from Eq. (57). The maximum value of the radial derivative occurs clearly on the wall of the sphere. Analysis of the resultant relation yields the following asymptotic estimates

$$q_2^2 \approx \begin{cases} \dfrac{\sin^2(\psi)}{\cos(\psi)} \cdot \dfrac{2B}{27}\,; & B \ll 1 \qquad (62a) \\[2ex] \dfrac{\sin^2(\psi)}{\cos^2(\psi)} \cdot \dfrac{1}{3^4} \cdot \left(\dfrac{\ln(3B)}{B}\right)^2; & B \gg 1 \qquad (62b) \end{cases}$$

The second equation was written on the assumption that $\cos(\psi) \approx 1$; it is thus valid only in the asymptotic region where the major increment of the resulting value of the integral (53) or (59b) occurs. Substituting these relations into Eq. (59b) there results

$$\Delta A(B) \approx \begin{cases} \dfrac{2}{81} B\,; & B \ll 1 \qquad (63a) \\[2ex] \dfrac{1}{27}\left(\dfrac{\ln(3B)}{3B}\right)^2; & B \gg 1\,. \qquad (63b) \end{cases}$$

In regions of applicability of the individual asymptotic expansions, *i.e.* roughly for $B \leqq 0{\cdot}5$ and $B \geqq 1{\cdot}5$, Eqs (63a, b) represent at most 0·5% differences between the upper and the lower bound. In practice it is usually more than a sufficient accuracy.

The problem of a rotating sphere in a Bingham plastic has been solved on the assumption of an infinite medium. It is apparent though that a sufficient condition is that the walls of the vessel be outside the region of the flow, *i.e.* that they satisfy the relation $\xi_W \geqq \xi_0(\psi)$. Having satisfied this assumption the results presented here for $B \ll 1$ are comparable with those of Malinin[12] reporting an expression analogous to Eq. (61a). The difference is only in the numerical coefficient of $B^{3/2}$ probably due to inadequately constructed approximation. From the presented estimate of the error in Eqs. (63a, b) it is clear that the assumption imposed by Malinin, which in our notation may be written as $\xi_0 \ll 1$, is redundant. It suffices that $\xi_W \gtreqless \xi_0$ for all ψ. If this is not the case the presented approach can be modified by replacing Eq. (54) for ξ_0 by

$$3B\,.\cos(\psi) = \exp(3\xi_0)\,.\,(1 - \exp(-3\xi_a)) - 3\xi_a\,, \qquad (64a)$$

where

$$\xi_a = \min(\xi_0, \xi_W)\,. \qquad (64b)$$

Equally as in the previous case $\xi_W \geqq \xi_0$, here too $\xi_W \ll 1$ and $\xi_0 \ll 1$ are unnecessary in contrast to Malinin's assumption[12]. However, rotational viscometers with the close spherical vessel cannot be regarded as really suitable for testing viscoplastic materials and for this reason this case shall not be dealt with in detail.

CONCLUSION

The presented paper was intended to be as an introduction. It formulates mathematical prerequisites already used for the study[12] of specific problems of creeping non-Newtonian flow with the aim to work out adequate routine of processing data from portable rotational viscometer. Nevertheless, it is felt that the possibility of formulating approximate analytical solutions for a nonlinear two-dimensional problem for a rather general class of geometrical configuration and rheological models deserves an independent communication.

The essence of this work has been the proof that with and orthogonal or conformal mapping which is regular on Γ_A, and Γ_M the one-gradient approximation satisfies all boundary conditions of the problem and offers both the upper and the lower bounds for the functional whose minimum is a point of the exact solution to the problem. For the power-law model the one-gradient solution provides the upper and the lower bound also for the value of the dissipation functional which can be viewed as the principal result. It may be expected that these estimates will be applicable also for other sufficiently smooth viscosity functions.

The paragraph dealing with the construction of the minimizing sequence has a rather illustrative character. As useful though appears the relation between δJ and δE applicable generally to othe rdirect methods of solution than the Ritz' method. The quasi-linearization (*43*) is, of course, based on the assumption of adequate representation of succesive differences by variations and its prerequisite is a good initial estimate. Whether the one-gradient approximation provides a sufficiently good initial guess even in nontrivial cases, when for instance at a certain point on $\mathscr{A}$ $g \to \infty$, will be shown by numerical experiments.

The example presented in the text suggests that in many cases the one-gradient estimate $E_1[\omega_1]$ is so good that it does not call for any further refinement. Also the study of rotational flow of a power-law liquid in a system of confocal spheroids[16] the one-gradient approximation turned out to give good entry estimates of the velocity field and the intensity of dissipation[16].

Thanks are due to Dr K. Wichterle for encouragement to undertake this work and to Dr P. Mitschka for critical discussions.

LIST OF SYMBOLS

$\mathscr{A}$	studied region of flow
A	dimensionless torque, Eq. (*53*)
B	dimensionless angular velocity, Eq. (*55*)
$C(\psi)$	parameter of the one-gradient approximation, Eqs (*25*), (*26*)
D	shear rate, Eq. (*11*)
D_{ij}	components of rate of deformation tensor, Eqs (*8a,b*)
E	dissipation functional, Eq. (*31*)
g	normalized Lamé's coefficient, Eq. (*9*)
J	principal functional of the problem, Eqs (*7*), (*10*)
J_1	principal functional of the one-gradient approximation, Eq. (*21*)
M	torque
n'	local flow index, Eq. (*6b*)
N, P	auxiliary quantities, Eqs (*16a,b*).
Q	dissipation
r	polar radius
S	auxiliary quantity, Eq. (*16c*)
U	auxiliary energy function, Eq. (*6a*)
v_i	physical components of velocity
w	dimensionless angular velocity, Eq. (*52*)
z	axial polar coordinate
Γ	boundaries of the studied flow region
ε	variation of angular velocity; function on $\mathscr{A}$ satisfying conditions (*18*)
$\{\varepsilon\}$	linear space of all continuously differentiable functions with the limited norm $\|\varepsilon\|_{\omega e}$
η	viscosity function, Eq. (*1*)
μ_B	plastic viscosity
ξ	natural coordinate
ξ_0	boundary of flow region
τ	shear stress
τ_{ij}	components of tensor of deformation stress
τ_0	yield stress
ϕ	angular coordinate
ψ	natural longitudinal coordinate
ω	angular velocity; function on A satisfying conditions (*5*)
$\{\omega\}$	linear space of all a ω with limited norm $\|\omega\|_{\omega e}$
ω^1	one-gradient approximation, Eq. (*25*)
Ω	angular velocity of spindle

Subscript

A	symmetry axis
e	exact solution
M	plane of symmetry
R	surface of rotating spindle
W	surface of wall

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